Review for Midterm1 Test
(5.5, 6.1, 6.2, 6.3, 7.1, 7.2)

5.5

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}}\, dx$$

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}}\, dx$$

Substitute $\sqrt{x} = u$ and $\frac{1}{2\sqrt{x}}\, dx = du$

$$= 2\int (\sin \sqrt{x}) \left(\frac{1}{2\sqrt{x}}\, dx\right)$$

$$= 2\int \sin u\, du$$

$$= -2\cos u + C$$

Substitute back $u = \sqrt{x}$

$$= -2\cos \sqrt{x} + C$$

$$\int \sin x \sin(\cos x)\, dx$$

We can rewrite the given integral as:

$$\int \sin(\cos x) \sin x\, dx$$

$\text{We will substitute } \cos x = u \text{ And on differentiating we get } -\sin x\, dx = du \implies \sin x\, dx = -du$

$$= \int \sin u \cdot (-du)$$

$$= \int -\sin u\, du$$

$$= \cos u + C$$

Substitute back $u = \cos x$

$$= \cos(\cos x) + C$$

$$h(x) = \int_1^{e^x} \ln t\, dt$$

. Calculate the first derivative

Given that

$$h(x) = \int_1^{e^x} \ln t\; dt$$

Therefore

$$h'(x) = \frac{d\left[\int_1^{e^x} \ln t\; dt\right]}{dx}$$

Use chain rule

$$h'(x) = \frac{d\left[\int_1^{e^x} \ln t\; dt\right]}{d\left(e^x\right)} \cdot \frac{d\left(e^x\right)}{dx}$$

$$h'(x) = \ln e^x \cdot e^x$$

$$\text{Recall that: } \ln e^x = x$$

$$h'(x) = xe^x$$

$$h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4 + 1}\, dz$$

. Calculate the first derivative

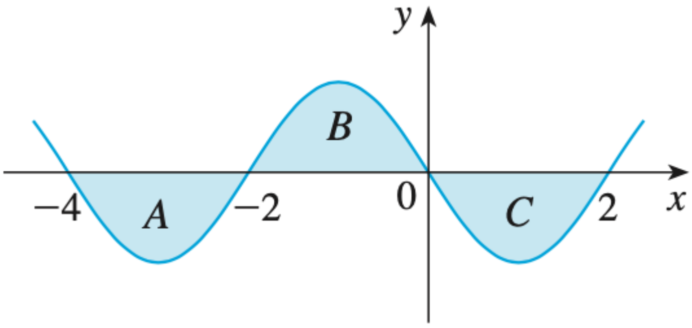
$$h(x) = \left[\frac{\left(\sqrt{x}\right)^2}{\left(\sqrt{x}\right)^4 + 1}\right] \cdot \left[\frac{1}{2} \cdot x^{(1/2)-1}\right]$$

$$h(x) = \frac{\left(\sqrt{x}\right)^2}{x^2 + 1} \cdot \frac{1}{2\sqrt{x}}$$

$$h(x) = \frac{\sqrt{x}}{2x^2 + 2}$$

Each of the regions A, B, and C bounded by the graph of f and the x-axis has area 3. Find the value of

$$\int_{-4}^{2} [f(x) + 2x + 5]\, dx$$

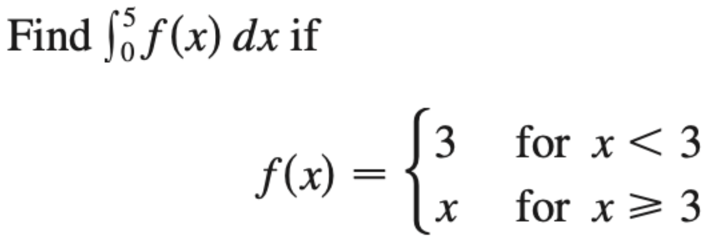


$$\int_{-4}^{2} f(x)\, dx = -3 + 3 - 3 = -3$$

$$\int_{-4}^{2} [f(x) + 2x + 5]\, dx = \int_{-4}^{2} f(x)\, dx + \int_{-4}^{2} 2x\, dx + \int_{-4}^{2} 5\, dx$$

$$= -3 - 12 + 30$$

$$= 15$$



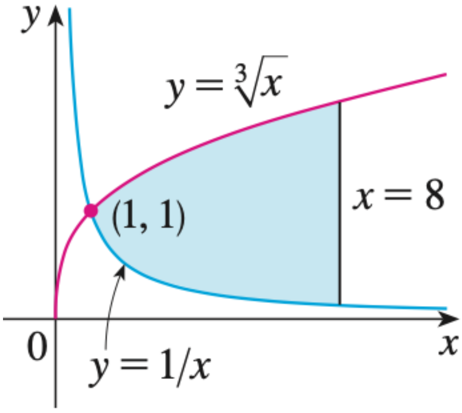
Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geqslant 3 \end{cases}$$

$$\int_0^5 f(x)dx = \int_0^3 f(x)dx + \int_3^5 f(x)dx = \int_0^3 3dx + \int_3^5 xdx = 9 + 8 = 17$$

Notice the integral represents the total area of a rectangle $\displaystyle\int_0^3 3dx$ and a trapezoid $\int_3^5 xdx$. The rectangle $\displaystyle\int_0^3 3dx$ has base = 3 and height = 3, thus its area is $A = 3(3) = 9$, and the trapezoid $\int_3^5 xdx$ has bases = 3 and 5 and height = 2, thus its area is $A = \frac{1}{2}(3+5)(2) = 8$. (See graph below.)

6.1



$$\int_1^8 \sqrt[3]{x} - \frac{1}{x}\,dx$$

$$= \int_1^8 x^{1/3} - \frac{1}{x}\,dx$$

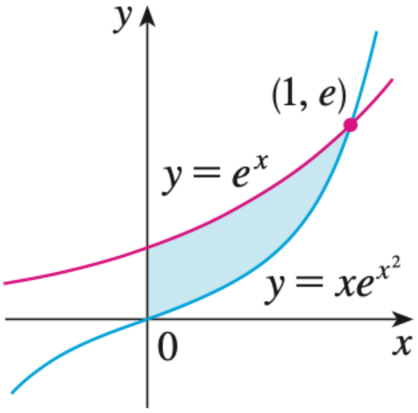
$$= \left[\frac{3}{4}x^{4/3} - \ln x\right]_1^8$$

$$= \left[\frac{3}{4}x\sqrt[3]{x} - \ln x\right]_1^8$$

$$= \left[\frac{3}{4}\cdot 8\sqrt[3]{8} - \ln 8\right] - \left[\frac{3}{4}\cdot 1\sqrt[3]{1} - \ln 1\right]$$

$$= \frac{48}{4} - \ln 8 - \frac{3}{4}$$

$$= \frac{45}{4} - \ln 8$$



$$\int_0^1 e^x - xe^{x^2}\,\mathrm{d}x \tag{1}$$

Now lets separe it into 2 integrals and solve them:

$$\int_0^1 e^x\,\mathrm{d}x - \int_0^1 xe^{x^2}\,\mathrm{d}x$$

$$\int_0^1 e^x\,\mathrm{d}x = e^x\Big|_0^1 = e - 1$$

For the other integral we will use the following substitution:

$$u = x^2\,\mathrm{d}x \implies \mathrm{d}u = 2x\,\mathrm{d}x \implies \frac{\mathrm{d}u}{2x} = \mathrm{d}x$$

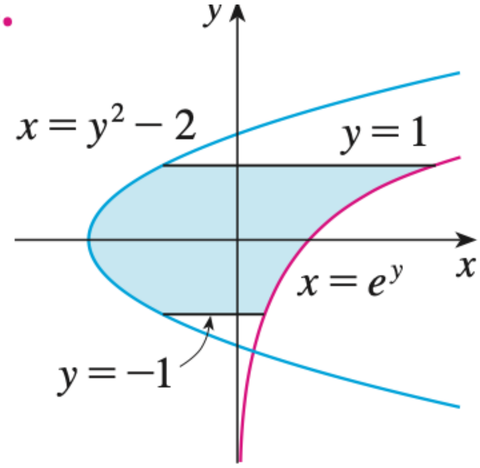
Now we have:

$$\int_0^1 xe^u \frac{1}{2x}\,\mathrm{d}u$$

$$\frac{1}{2}\int_0^1 e^u = \frac{1}{2}e^u\Big|_0^1 = \frac{e}{2} - \frac{1}{2}$$

Thus we can conclude that the result of (1) is as follows:

$$\int_0^1 e^x - xe^{x^2}\,\mathrm{d}x = e - 1 - \left(\frac{e}{2} - \frac{1}{2}\right) = \frac{1}{2}(e-1)$$



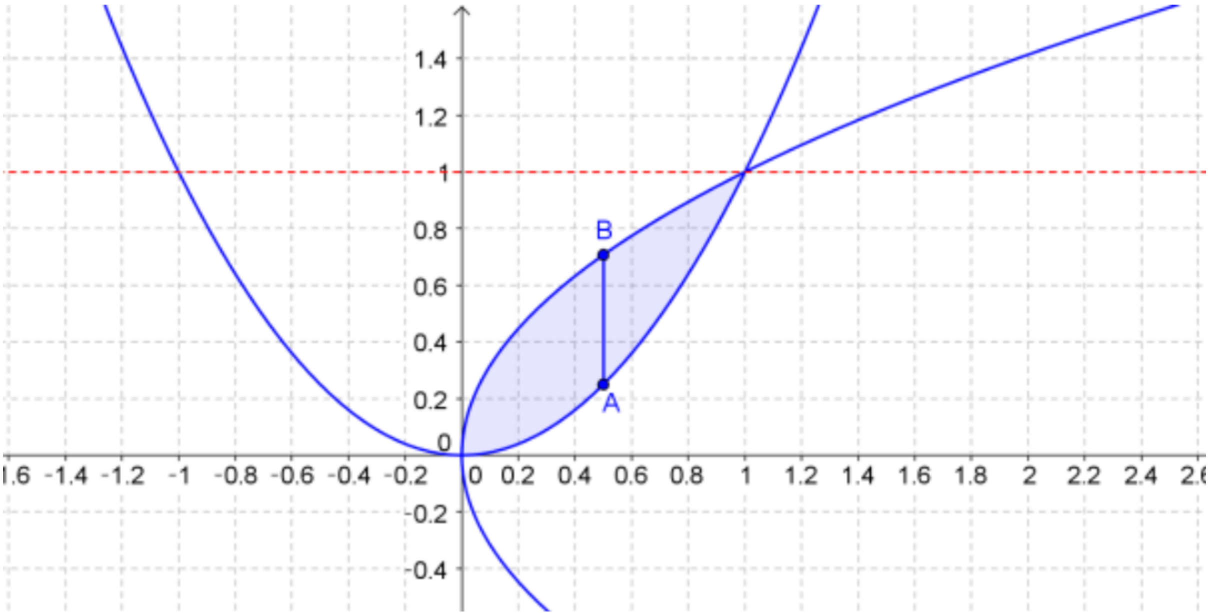
can write (2) as:

$$
\begin{aligned}
A &= \int_{-1}^{1} \left[e^y - \left(y^2 - 2\right)\right] dy \\
&= \int_{-1}^{1} \left(e^y - y^2 + 2\right) dy \\
&= \int_{-1}^{1} e^y dy - \int_{-1}^{1} y^2 dy + 2 \int_{-1}^{1} dy
\end{aligned}
$$

$$
\begin{aligned}
A &= \int_{-1}^{1} \left[e^y - \left(y^2 - 2\right)\right] dy \\
&= \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^{1} \\
&= \left(e^1 - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right) \\
&= e - \frac{1}{e} + \frac{10}{3}
\end{aligned}
$$

6.2

$y = x^2, \; x = y^2; \quad \text{about } y = 1$



$$= \pi \int_0^1 ((1 - x^2)^2 - (1 - \sqrt{x})^2)\, dx$$

$$= \pi \int_0^1 (1 - 2x^2 + x^4 - (1 - 2\sqrt{x} + x))\, dx$$

$$= \pi \int_0^1 (1 - 2x^2 + x^4 - 1 + 2\sqrt{x} - x)\, dx$$

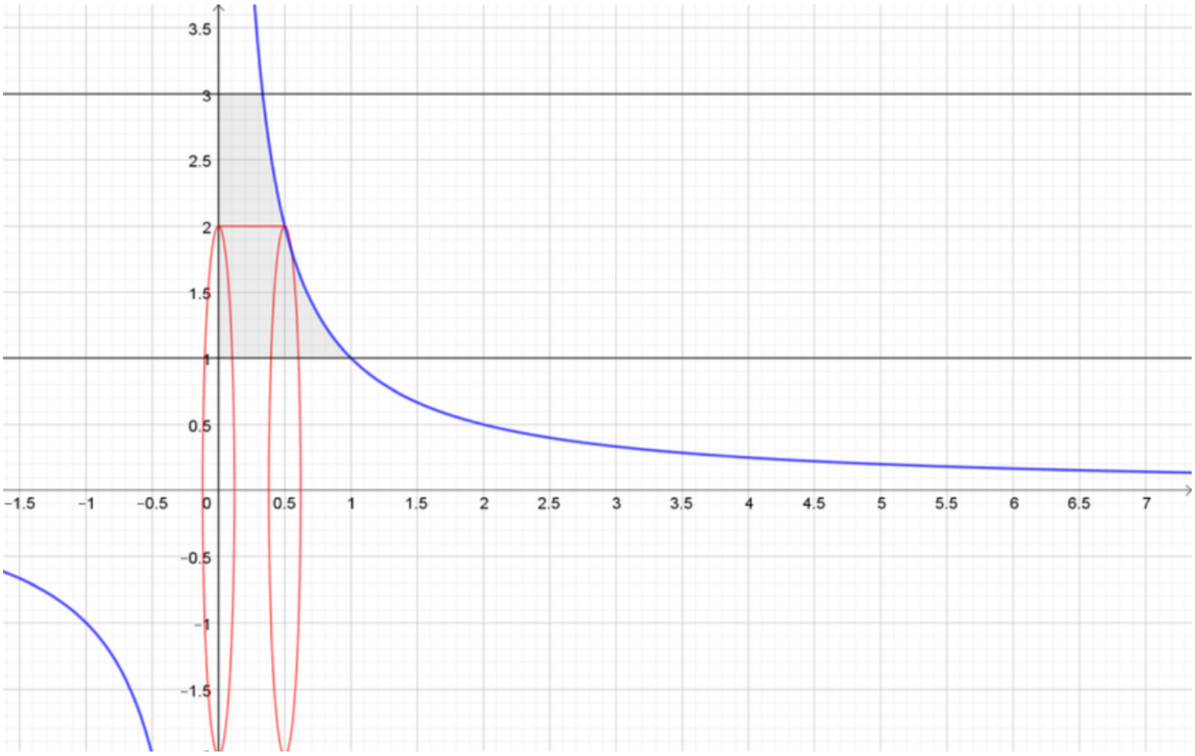
$$= \pi \int_0^1 (-2x^2 + x^4 + 2\sqrt{x} - x)\, dx$$

$$= \frac{11\pi}{30}$$

6.3

$$xy = 1, \quad x = 0, \quad y = 1, \quad y = 3$$

about x axis



$$V = \int_a^b 2\pi r h \, dy = 2\pi \int_1^3 y \cdot \frac{1}{y} \, dy = 2\pi \int_1^3 dy$$

$$= 2\pi \left[y\right]_1^3 = 2\pi(3-1) = 4\pi$$

7.1

$$\int (\ln x)^2\, dx$$

Let $u = (\ln x)^2$, $dv = dx$

Then $du = \frac{2\ln x}{x} dx$, $v = x$

We know that $\int u\; dv = u\; v - \int v\; du$

Integration by parts gives

$$I = x(lnx)^2 - \int x.\frac{2\ln x}{x} dx$$

$$I = x(lnx)^2 - 2\int \ln x dx$$

Step 2

Find $I_2 = \int \ln x dx$

Let $u = \ln x$, $dv = dx$

Then $du = \frac{1}{x} dx$, $v = x$

We know that $\int u\; dv = u\; v - \int v\; du$

Integration by parts gives

$$I_2 = x\ln x - \int x.\frac{1}{x} dx$$

$$I_2 = x\ln x - \int dx$$

$$I_2 = x\ln x - x + c$$

$$I = x(lnx)^2 - 2(x\ln x - x + c)$$

$$= x(lnx)^2 - 2x\ln x + 2x + C$$

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

$$\begin{aligned}
I &= \overbrace{\int \frac{xe^{2x}}{(1+2x)^2}\,dx}^{\boxed{\begin{array}{c}\textbf{Integration by parts}\\ u=xe^{2x} \quad du=e^{2x}+2xe^{2x}=e^{2x}(1+2x)\\ dv=\frac{1}{(1+2x)^2} \quad v=\frac{-1}{2(1+2x)}\end{array}}} \\
&= xe^{2x}\cdot\frac{-1}{2(1+2x)} - \int e^{2x}(1+2x)\cdot\frac{-1}{2(1+2x)}\,dx \\
&= -\frac{xe^{2x}}{2(1+2x)} - \int \frac{-1}{2}e^{2x}\,dx \\
&= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int e^{2x}\,dx \\
&= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C \\
&= -\frac{xe^{2x}}{2(1+2x)} + \frac{e^{2x}}{4} + C \\
&= -\frac{2xe^{2x}}{4(1+2x)} + \frac{(1+2x)e^{2x}}{4(1+2x)} + C \\
&= \boxed{\frac{e^{2x}}{4(1+2x)} + C}
\end{aligned}$$

7.2

$\sin^3\theta\ \cos^4\theta\ d\theta$

$$\int \sin^3\theta\cos^4\theta\, d\theta$$

$$= \int \sin^2\theta\cos^4\theta \cdot (\sin\theta\, d\theta)$$

$$= \int \left(1-\cos^2\theta\right)\cos^4\theta \cdot (\sin\theta\, d\theta)$$

$$= \int \left(\cos^2\theta - 1\right)\cos^4\theta \cdot (-\sin\theta\, d\theta)$$

Substitute $\cos\theta = u$

And $-\sin\theta\, d\theta = du$

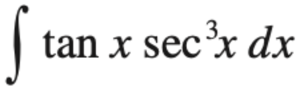
$$= \int \left(u^2-1\right)u^4\, du$$

$$= \int u^6 - u^4\, du$$

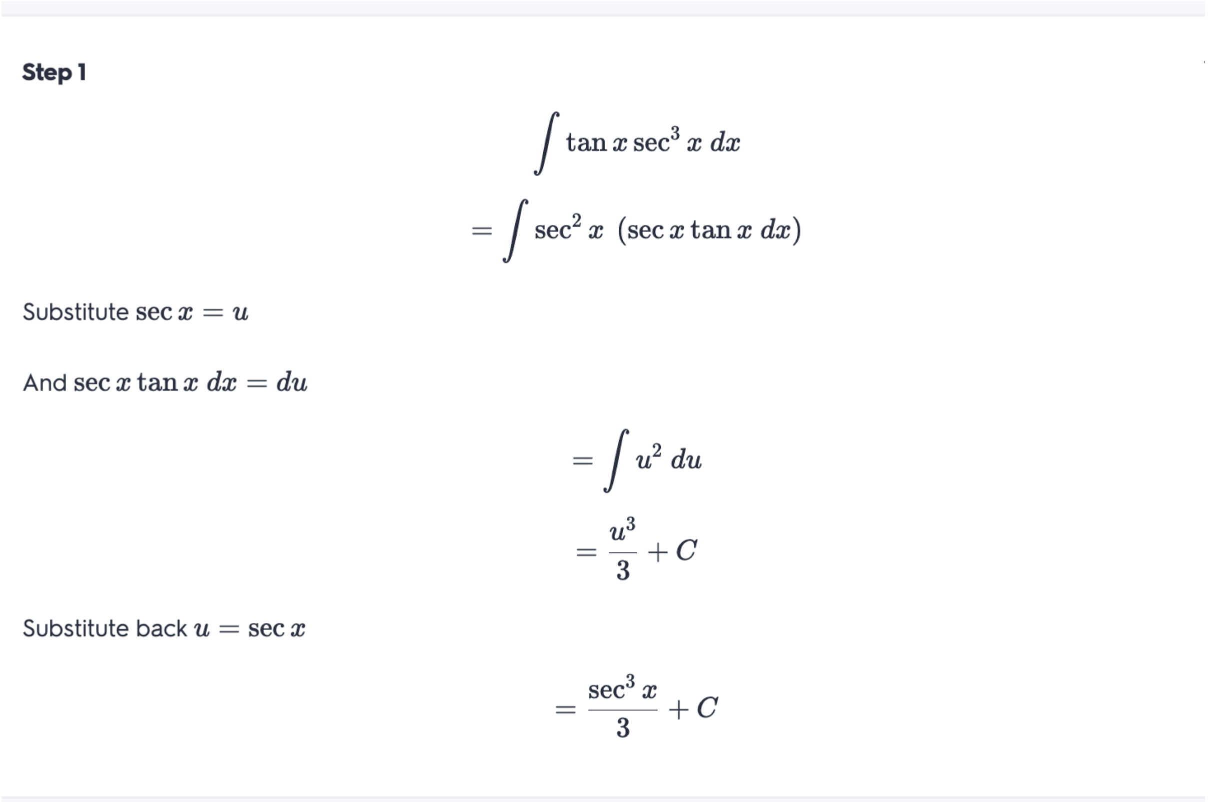
$$= \frac{u^7}{7} - \frac{u^5}{5} + C$$

Substitute back : $u = \cos\theta$

$$= \frac{\cos^7\theta}{7} - \frac{\cos^5\theta}{5} + C$$



$$\int \tan x \sec^3 x \, dx$$



Step 1

$$\int \tan x \sec^3 x \; dx$$

$$= \int \sec^2 x \;\; (\sec x \tan x \; dx)$$

Substitute $\sec x = u$

And $\sec x \tan x \; dx = du$

$$= \int u^2 \; du$$

$$= \frac{u^3}{3} + C$$

Substitute back $u = \sec x$

$$= \frac{\sec^3 x}{3} + C$$

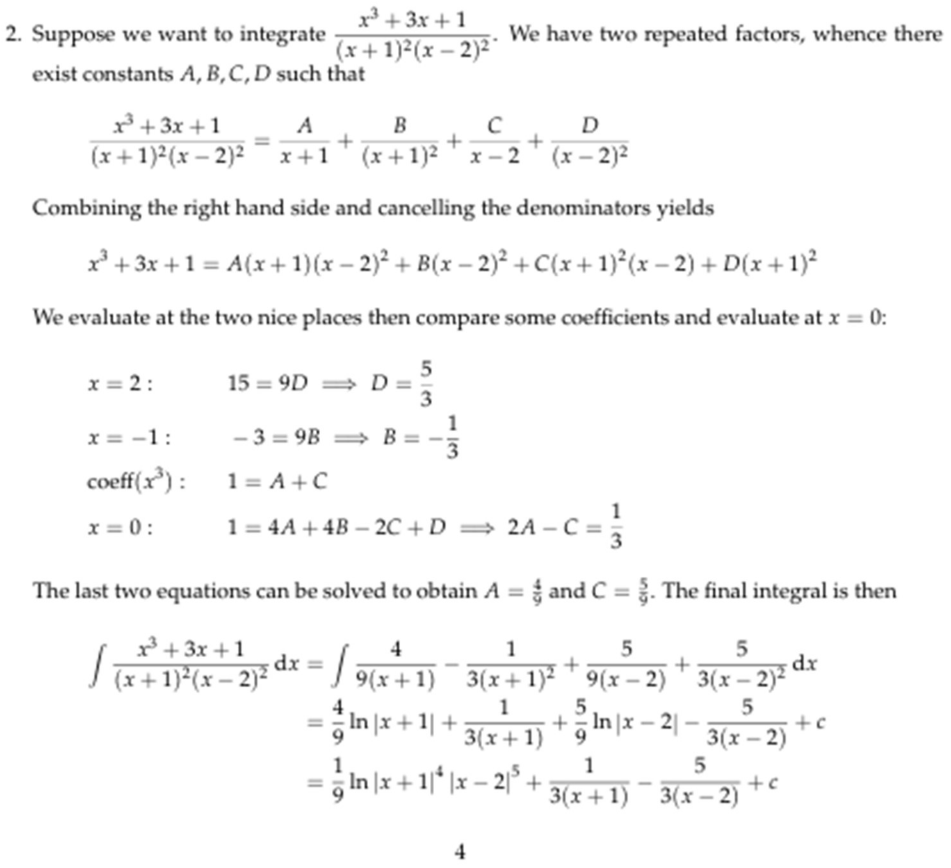
<u>STUDY</u>

CLASS NOTES
SLIDES
QUIZ 1
PROBLEMS IN THE SYLLABUS

GOOD LUCK

REVIEW FOR MIDTERM2 EXAM

7.4



2. Suppose we want to integrate $\frac{x^3+3x+1}{(x+1)^2(x-2)^2}$. We have two repeated factors, whence there exist constants A, B, C, D such that

$$\frac{x^3+3x+1}{(x+1)^2(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

Combining the right hand side and cancelling the denominators yields

$$x^3+3x+1 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2) + D(x+1)^2$$

We evaluate at the two nice places then compare some coefficients and evaluate at $x = 0$:

$$\begin{aligned}
x = 2: &\quad 15 = 9D \implies D = \frac{5}{3} \\
x = -1: &\quad -3 = 9B \implies B = -\frac{1}{3} \\
\text{coeff}(x^3): &\quad 1 = A + C \\
x = 0: &\quad 1 = 4A + 4B - 2C + D \implies 2A - C = \frac{1}{3}
\end{aligned}$$

The last two equations can be solved to obtain $A = \frac{4}{9}$ and $C = \frac{5}{9}$. The final integral is then

$$\begin{aligned}
\int \frac{x^3+3x+1}{(x+1)^2(x-2)^2}\,dx &= \int \frac{4}{9(x+1)} - \frac{1}{3(x+1)^2} + \frac{5}{9(x-2)} + \frac{5}{3(x-2)^2}\,dx \\
&= \frac{4}{9}\ln|x+1| + \frac{1}{3(x+1)} + \frac{5}{9}\ln|x-2| - \frac{5}{3(x-2)} + c \\
&= \frac{1}{9}\ln|x+1|^4\,|x-2|^5 + \frac{1}{3(x+1)} - \frac{5}{3(x-2)} + c
\end{aligned}$$

Example The rational function $\frac{x^2-x+2}{x^3+4x} = \frac{x^2-x+2}{x(x^2+4)}$ contains the irreduciuble quadratic x^2+4 in its denominator. We therefore know that there exist constants A, B, C such that

$$\frac{x^2-x+2}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Combining the right hand side and equating numerators yields

$$x^2-x+2 = A(x^2+4) + (Bx+C)x$$

which can be solved (try it!) to obtain

$$A = \frac{1}{2}, \quad B = \frac{1}{2}, \quad C = -1$$

It follows that

$$\begin{aligned}\int \frac{x^2-x+2}{x^3+4x}\,\mathrm{d}x &= \int \frac{1}{2x} + \frac{x-2}{2(x^2+4)}\,\mathrm{d}x = \frac{1}{2}\ln|x| + \int \frac{x}{2(x^2+4)} - \frac{1}{x^2+4}\,\mathrm{d}x \\ &= \frac{1}{2}\ln|x| + \frac{1}{4}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + c\end{aligned}$$

Rationalizing

A clever substitution can sometimes convert an irrational expression into a rational one, to which the partial fractions method may be applied.
For example, the substitution $u^3 = x-7$ $(\mathrm{d}x = 3u^2\,\mathrm{d}u)$ gives

$$\begin{aligned}\int \frac{\sqrt[3]{x-7}}{x+1}\,\mathrm{d}x &= \int \frac{3u^3}{u^3+8}\,\mathrm{d}u = \int 3 - \frac{24}{(u+2)(u^2-2u+4)}\,\mathrm{d}u \\ &= 3u + \ln\frac{u^2-2u+4}{(u+2)^2} - 2\sqrt{3}\tan^{-1}\frac{u-1}{\sqrt{3}} + c \qquad \text{(partial fractions in here)} \\ &= 3(x-7)^{1/3} + \ln\frac{(x-7)^{2/3} - 2(x-7)^{1/3} + 4}{((x-7)^{1/3}+2)^2} - 2\sqrt{3}\tan^{-1}\frac{(x-7)^{1/3}-1}{\sqrt{3}} + c\end{aligned}$$

A similar approach (substituting $u = \sqrt{x-2}$) rationalizes the integral

$$\int \frac{1}{(x-2)(x-2+\sqrt{x-2})}\,\mathrm{d}x = \int \frac{2\,\mathrm{d}u}{u^2(u+1)}$$

7.8

A. TYPE 2

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^-}\int_t^1 x^{1/2}\,dx = \lim_{t\to 0^-}\left[2x^{1/2}\right]_t^1 = 2\lim_{t\to 0^-}\left[1 - t^{1/2}\right] = 2$$

B. Using $u = x^2$: TYPE 1

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^{t^2} e^{-u}\frac{du}{2} = \frac{1}{2}\lim_{t\to\infty}\left[1 - e^{-t^2}\right] = \frac{1}{2}$$

C. DIVERGES, TYPE 1

$$\int_{-\infty}^{-1}\frac{dx}{x} = \lim_{t\to-\infty}\int_t^{-1}\frac{dx}{x} = \lim_{t\to-\infty}\left[\ln|x|\right]_t^{-1} = \lim_{t\to-\infty}\ln 1 - \ln(-t) = -\infty$$

D. DIVERGES, TYPE 1

$$\int_0^\infty e^x\,dx = \lim_{t\to\infty}\int_0^t e^x\,dx = \lim_{t\to\infty}\left[e^x\right]_0^t = \lim_{t\to\infty} e^t - 1 = +\infty$$

E. Use $u = 5 - x$ to clarify which limit is improper: TYPE 2

$$\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx = \int_5^0 u^{-1/3}\,(-du) = \int_0^5 u^{-1/3}\,du = \lim_{t\to 0^+}\int_t^5 u^{-1/3}\,du = \lim_{t\to 0^+}\left[\frac{3}{2}u^{2/3}\right]_t^5$$

$$= \frac{3}{2}\lim_{t\to 0^+}(5^{2/3} - t^{2/3}) = \frac{3}{2}5^{2/3}$$

F. Split at discontinuity at $c = 2$. Use $u = w - 2$ to show one integral diverges. Thus the original integral diverges.

DIVERGES, TYPE 2

$$\int_0^5 \frac{w}{w-2}\,dw = \int_0^2 \frac{w}{w-2}\,dw + \int_2^5 \frac{w}{w-2}\,dw$$

and

$$\int_0^2 \frac{w}{w-2}\,dw = \int_{-2}^0 \frac{u+2}{u}\,du = 2 + 2\lim_{t\to 0^-}\int_{-2}^t \frac{du}{u}$$

$$= 2 + 2\lim_{t\to 0^-}\left[\ln|u|\right]_{-2}^t = 2 + 2\lim_{t\to 0^-}(\ln(-t) - \ln 2) = -\infty$$

G. TYPE 1

$$\int_{-\infty}^0 2^r\,dr = \lim_{t\to-\infty}\left[\frac{2^r}{\ln 2}\right]_t^0 = \frac{1}{\ln 2}\lim_{t\to-\infty} 1 - 2^t = \frac{1}{\ln 2}(1 - 0) = \frac{1}{\ln 2}$$

H. Use $u = \sqrt{y}$ and then integrate by parts with $w = u$ and $dv = e^{-u}\,du$: TYPE 1

$$\int_0^\infty e^{-\sqrt{y}}\,dy = \lim_{t\to\infty}\int_0^t e^{-\sqrt{y}}\,dy = \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-u}(2u)\,du = 2\lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u}\,du$$

$$= 2\lim_{t\to\infty}\left(\Big[-ue^{-u}\Big]_0^{\sqrt{t}} + \int_0^{\sqrt{t}} e^{-u}\,du\right) = 2\lim_{t\to\infty}\left(-\sqrt{t}e^{-\sqrt{t}} + \left[-e^{-u}\right]_0^{\sqrt{t}}\right)$$

$$= 2\lim_{t\to\infty}\left(-\sqrt{t}e^{-\sqrt{t}} - e^{-\sqrt{t}} + 1\right) = 2(0+0+1) = 2$$

I. Split at discontinuity $x = 1$ and compute each improper integral using $u = x - 1$: TYPE 2

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = -\frac{3}{2} + 6 = \frac{9}{2}$$

because

$$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_{-1}^0 u^{-1/3}\,du = \lim_{t\to0^-}\left[\frac{3}{2}u^{2/3}\right]_{-1}^t = \frac{3}{2}\lim_{t\to0^-}\left(t^{2/3} - 1\right) = -\frac{3}{2}$$

and

$$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^8 u^{-1/3}\,du = \lim_{t\to0^+}\left[\frac{3}{2}u^{2/3}\right]_t^8 = \frac{3}{2}\lim_{t\to0^+}\left(8^{2/3} - t^{2/3}\right) = \frac{3}{2}(4-0) = 6$$

7. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$

Solution:

(a) Improper because it is an infinite integral (called a Type I).

(b) Let's do a u-substitution first. Let $u = e^x$, then $du = e^x dx$. When $x = 0, u = 1$ and when $x \to \infty, u \to \infty$:

$$\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx = \int_0^\infty \frac{e^x}{(e^x)^2+3}\,dx = \int_1^\infty \frac{1}{u^2+3}\,du = \lim_{t\to\infty}\int_1^t \frac{1}{u^2+3}\,du$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Big|_1^t = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

Convergent! □

8.1

Arc Length

1. Calculate the length of the following lines using the arc length calculation formula $\left(\ell = \int \sqrt{1 + (f'(x))^2} dx\right)$. Compare the results to the geometric calculation of the length of the line.

   (a) The line $y = 0$ between $x = 0$ and $x = 1$

$$y = f(x) = 0$$

so

$$f'(x) = 0$$

Then, the arclength is

$$\ell = \int_{x=0}^{x=1} \sqrt{1+0} dx = \int_{x=0}^{x=1} 1 dx$$

The antiderivative of 1 is x, so the value is

$$\ell = [x]_0^1 = 1 - 0 = 1$$

The length of the line is $\boxed{1}$. This is the same result as the geometric result: that the length of the line between $(0, 0)$ and $(1, 0)$ is clearly 1.

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(c) The line $y = x$ between $x = 0$ and $x = 2$

1(c). **Answer:** $\boxed{2\sqrt{2}}$. For the line $y = f(x) = x$, the derivative $f'(x)$ is not equal to zero. Instead,

$$f'(x) = 1$$

Then, we compute

$$\ell = \int_{x=0}^{x=2} \sqrt{1 + (f'(x))^2} dx = \int_{x=0}^{x=2} \sqrt{1+1} dx$$

which is just

$$\sqrt{2} \int_{x=0}^{x=2} 1 dx = \sqrt{2}(2-0) = 2\sqrt{2}$$

==

(f) The line $y = ax + b$ between $x = x_1$ and $x = x_2$, assuming $x_2 > x_1$.

1(f). **Answer:** $\boxed{\sqrt{1+a^2}\,(x_2 - x_1)}$. For the line $y = f(x) = ax + b$ the derivative $f'(x)$ is equal to

$$f'(x) = a$$

Then, we compute

$$\ell = \int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}dx = \int_{x_1}^{x_2} \sqrt{1+a^2}dx = \sqrt{1+a^2}\,[x]_{x_1}^{x_2}$$

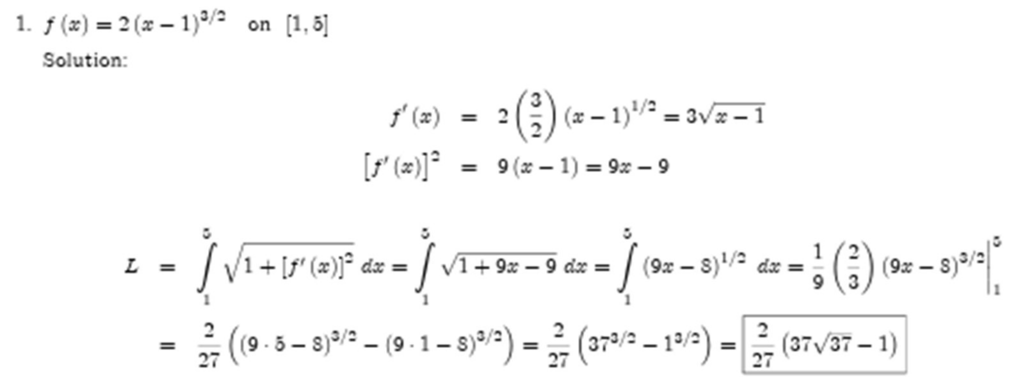
which is $\boxed{\sqrt{1+a^2}\,(x_2 - x_1)}$. This is the same result that we can derive geometrically: the distance between

$$(x_1, ax_1 + b) \to (x_2, ax_2 + b)$$

is

$$\sqrt{(x_2 - x_1)^2 + (ax_2 + b - ax_1 - b)^2} = (x_2 - x_1)\sqrt{1+a^2}$$

.



1. $f(x) = 2(x-1)^{3/2}$ on $[1, 5]$

Solution:

$$\begin{aligned} f'(x) &= 2\left(\frac{3}{2}\right)(x-1)^{1/2} = 3\sqrt{x-1} \\ \left[f'(x)\right]^2 &= 9(x-1) = 9x - 9 \end{aligned}$$

$$\begin{aligned} L &= \int_1^5 \sqrt{1 + [f'(x)]^2}\,dx = \int_1^5 \sqrt{1 + 9x - 9}\,dx = \int_1^5 (9x-8)^{1/2}\,dx = \frac{1}{9}\left(\frac{2}{3}\right)(9x-8)^{3/2}\Big|_1^5 \\ &= \frac{2}{27}\left((9\cdot 5 - 8)^{3/2} - (9\cdot 1 - 8)^{3/2}\right) = \frac{2}{27}\left(37^{3/2} - 1^{3/2}\right) = \boxed{\frac{2}{27}\left(37\sqrt{37} - 1\right)} \end{aligned}$$

4. $f(x) = \ln(\cos x)$ on $\left[0, \frac{\pi}{4}\right]$

Solution:

$$\begin{aligned} f(x) &= \ln(\cos x) \\ f'(x) &= \frac{1}{\cos x}(-\sin x) = -\tan x \\ \left[f'(x)\right]^2 + 1 &= \tan^2 x + 1 = \sec^2 x \end{aligned}$$

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{1 + [f'(x)]^2}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x|\Big|_0^{\pi/4} \\ &= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| = \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| == \ln\left(\sqrt{2} + 1\right) - \ln 1 \\ &= \boxed{\ln\left(\sqrt{2} + 1\right)} \end{aligned}$$

---

3. $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ on $[1, 3]$

Solution:

$$\begin{aligned} f(x) &= \frac{x^3}{6} + \frac{1}{2x} \\ f'(x) &= \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) \\ \left[f'(x)\right]^2 + 1 &= \frac{1}{4}\left(x^4 + \frac{1}{x^4} - 2\right) + 1 = \frac{1}{4}\left(x^4 + \frac{1}{x^4} - 2\right) + \frac{4}{4} = \frac{1}{4}\left(x^4 + \frac{1}{x^4} + 2\right) = \left[\frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)\right]^2 \end{aligned}$$

$$\begin{aligned} L &= \int_1^3 \sqrt{1 + [f'(x)]^2}\, dx = \int_1^3 \sqrt{\left[\frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)\right]^2}\, dx = \int_1^3 \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)\, dx = \frac{1}{2}\int_1^3 x^2 + \frac{1}{x^2}\, dx \\ &= \frac{1}{2}\left(\frac{x^3}{3} - \frac{1}{x}\right)\Big|_1^3 = \frac{1}{2}\left[\left(\frac{3^3}{3} - \frac{1}{3}\right) - \left(\frac{1^3}{3} - \frac{1}{1}\right)\right] = \frac{1}{2}\left[\left(9 - \frac{1}{3}\right) - \left(\frac{1}{3} - 1\right)\right] \\ &= \frac{1}{2}\left[\frac{26}{3} - \left(-\frac{2}{3}\right)\right] = \frac{1}{2}\left(\frac{28}{3}\right) = \boxed{\frac{14}{3}} \end{aligned}$$

8.2

2. $y = 1 - x^2, 0 \leq x \leq 1$, about the y-axis

   Solution: We are rotating about the y-axis so we want to use the formula $S = \int 2\pi x ds$. Since $0 \leq x \leq 1$, then we want to use $ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$.

   Find $\frac{dy}{dx}$: $\frac{dy}{dx} = -2x$

   Find ds: $ds = \sqrt{1 + (\frac{dy}{dx})^2} dx = \sqrt{1 + 4x^2} dx$

   Find S: $S = \int 2\pi x ds = \int_0^1 2\pi x \sqrt{1 + 4x^2} dx$

   Let $u = 1 + 4x^2$, then $du = 8x dx$. When $x = 0, u = 1$; when $x = 1, u = 5$:

$$S = \int_0^1 2\pi x \sqrt{1 + 4x^2} dx = \frac{2\pi}{8} \int_1^5 \sqrt{u}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^5 = \frac{\pi}{6}(5\sqrt{5} - 1)$$

□

---

5. $y = \sqrt{1 + e^x}, 0 \leq x \leq 1$, about the x-axis

   Solution: We are rotating about the x-axis so we want to use the formula $S = \int 2\pi y ds$. Since $0 \leq x \leq 1$, then we want to use $ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$.

   Find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{e^x}{2\sqrt{1 + e^x}}$$

   Find ds:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \frac{e^{2x}}{4(1 + e^x)}} dx = \sqrt{\frac{4 + 4e^x + e^{2x}}{4(1 + e^x)}} dx = \sqrt{\frac{(2 + e^x)^2}{4(1 + e^x)}} dx = \frac{2 + e^x}{2\sqrt{1 + e^x}} dx$$

   Find S:

$$S = \int 2\pi y ds = \int_0^1 2\pi \sqrt{1 + e^x} \frac{2 + e^x}{2\sqrt{1 + e^x}} dx = \pi \int_0^1 (2 + e^x) dx = \pi(2x + e^x)\Big|_0^1 = (e + 1)\pi$$

□

3. $9x = y^2 + 18, 2 \leq x \leq 6$, about the x-axis

   Solution: We are rotating about the x-axis so we want to use the formula $S = \int 2\pi y ds$. Since $2 \leq x \leq 6$, then we want to use $ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$.

First, solve for y in the above equation (there will be a ± solution, just pick the positive one):
$y = \sqrt{9x - 18} = 3\sqrt{x - 2}$
Find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{3}{2\sqrt{x-2}}$$

Find ds:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \frac{9}{4(x-2)}} dx = \sqrt{\frac{4(x-2)+9}{4(x-2)}} dx = \sqrt{\frac{4x+1}{4(x-2)}} dx = \frac{1}{2}\sqrt{\frac{4x+1}{x-2}} dx$$

Find S:

$$S = \int 2\pi y ds = \int_2^6 2\pi \cdot 3\sqrt{x-2} \cdot \frac{1}{2}\sqrt{\frac{4x+1}{x-2}} dx = 3\pi \int_2^6 \sqrt{4x+1}\ dx$$

Let $u = 4x + 1$, then $du = 4dx$. When $x = 2, u = 9$; when $x = 6, u = 25$:

$$= \frac{3\pi}{4} \int_9^{25} \sqrt{u}\ du = \frac{3\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_9^{25} = 49\pi$$

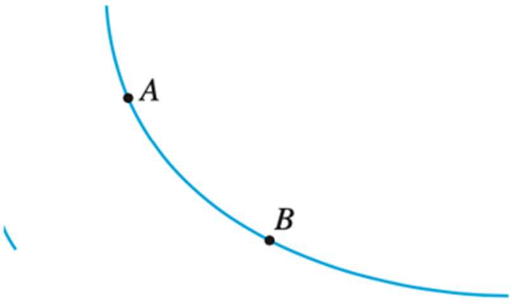
10.1

THEOREM Parametric Equations of a Cycloid

Parametric equations of a cycloid are

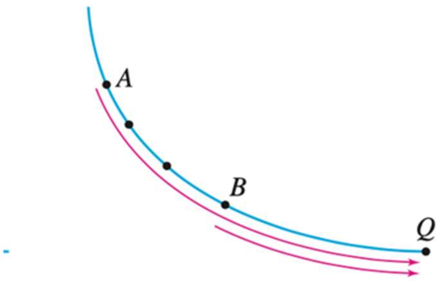
$$\boxed{x(t) = a(t - \sin t) \qquad y(t) = a(1 - \cos t) \qquad -\infty < t < \infty}$$

The **brachistochrone** is the curve of quickest descent. If an object is constrained to follow some path from a point A to a lower point B (not on the same vertical line) and is acted on only by gravity, the time needed to make the descent is minimized if the path is an inverted cycloid. See Figure 11(b). For example, in sliding packages from a loading dock onto a ship, a ramp in the shape of an inverted cycloid might be used so the packages get to the ship in the least amount of time. This discovery, which is attributed to many famous mathematicians (including Johann Bernoulli and Blaise Pascal), was a significant step in creating the branch of mathematics known as the *calculus of variations*.



(b) Curve of quickest descent

Suppose Q is the lowest point on an inverted cycloid. If several objects placed at various positions on an inverted cycloid simultaneously begin to slide down the cycloid, they will reach the point Q at the same time, as indicated in Figure 11(c). This is referred to as the **tautochrone property** of the cycloid. It was used by the Dutch mathematician, physicist, and astronomer Christian Huygens (1629–1695) to construct a pendulum clock



(c) All particles reach Q at the same time

(a) Find a rectangular equation of the plane curve whose parametric equations are

$$x(t) = \cos(2t) \qquad y(t) = \sin t \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Solution **(a)** To eliminate the parameter t, we use a trigonometric identity that involves $\sin t$ and $\cos(2t)$, namely, $\sin^2 t = \dfrac{1 - \cos(2t)}{2}$. Then

$$y^2 \underset{\substack{\uparrow \\ y(t) = \sin t}}{=} \sin^2 t = \frac{1 - \cos(2t)}{2} \underset{\substack{\uparrow \\ x(t) = \cos(2t)}}{=} \frac{1 - x}{2}$$

1. A curve C has parametric equations

$$x = 2t - 1, \quad y = 4t - 7 + \frac{3}{t}, \quad t \neq 0$$

Show that the Cartesian equation of the curve C can be written in the form

$$y = \frac{2x^2 + ax + b}{x + 1}, \quad x \neq -1$$

where a and b are integers to be found.

(3)

$C: x = 2t - 1$, $y = 4t - 7 + \frac{3}{t} \Rightarrow y = 4\left(\frac{x+1}{2}\right) - 7 + \frac{3 \times 2}{x+1}$

$x + 1 = 2t$

$\Rightarrow t = \frac{x+1}{2}$

$$\Rightarrow y = \frac{4(x+1)}{2} - 7 + \frac{6}{x+1} \quad ①$$

$$\Rightarrow y = 2x + 2 - 7 + \frac{6}{x+1}$$

$$\Rightarrow y = \frac{(2x - 5)}{1} + \frac{6}{x+1}$$

$$\Rightarrow y = \frac{(2x-5)(x+1) + 6}{x+1}$$

$2x - 5x = -3x$

$$\Rightarrow y = \frac{2x^2 - 3x - 5 + 6}{x+1} = \frac{2x^2 - 3x + 1}{x+1} \quad ①$$

$$y = \frac{2x^2 - 3x + 1}{x+1}$$, $a = -3$ and $b = 1$

2. A curve C has parametric equations

$$x = 3 + 2\sin t, \quad y = 4 + 2\cos 2t, \quad 0 \leqslant t < 2\pi$$

(a) Show that all points on C satisfy $y = 6 - (x - 3)^2$ → Cartesian

(2

(3)

a)

$\cos(2t) = 1 - 2\sin^2 t$

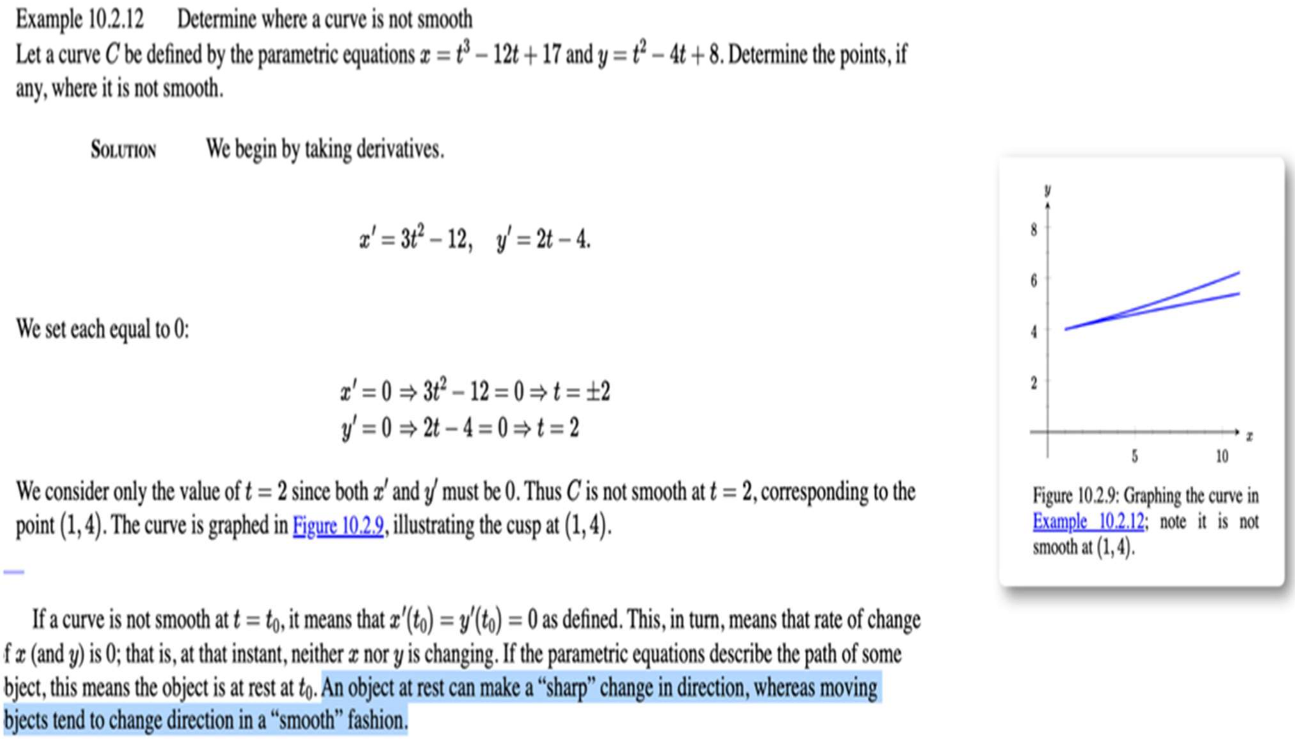
$y = 4 + 2(1 - 2\sin^2 t)$ ✓ $\quad x = 3 + 2\sin t$

$= 4 + 2 - 4\sin^2 t$ $\quad 2\sin t = x - 3$

$y = 6 - (2\sin t)^2$

$\therefore y = 6 - (x - 3)^2$ as required. ✓

10.2



Example 10.2.12 Determine where a curve is not smooth

Let a curve C be defined by the parametric equations $x = t^3 - 12t + 17$ and $y = t^2 - 4t + 8$. Determine the points, if any, where it is not smooth.

Solution We begin by taking derivatives.

$$x' = 3t^2 - 12, \quad y' = 2t - 4.$$

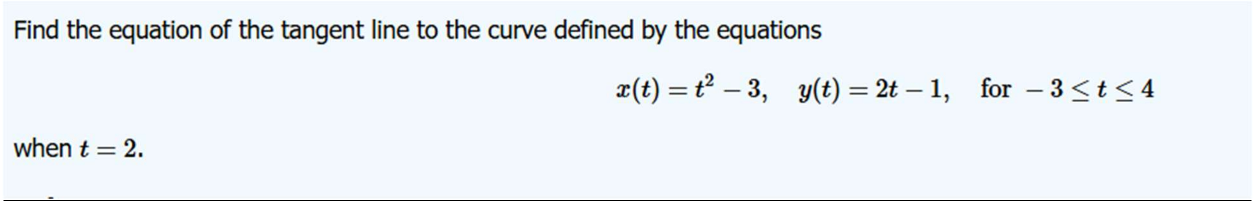
We set each equal to 0:

$$x' = 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow t = \pm 2$$
$$y' = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2$$

We consider only the value of $t = 2$ since both x' and y' must be 0. Thus C is not smooth at $t = 2$, corresponding to the point $(1, 4)$. The curve is graphed in Figure 10.2.9, illustrating the cusp at $(1, 4)$.

Figure 10.2.9: Graphing the curve in Example 10.2.12; note it is not smooth at $(1, 4)$.

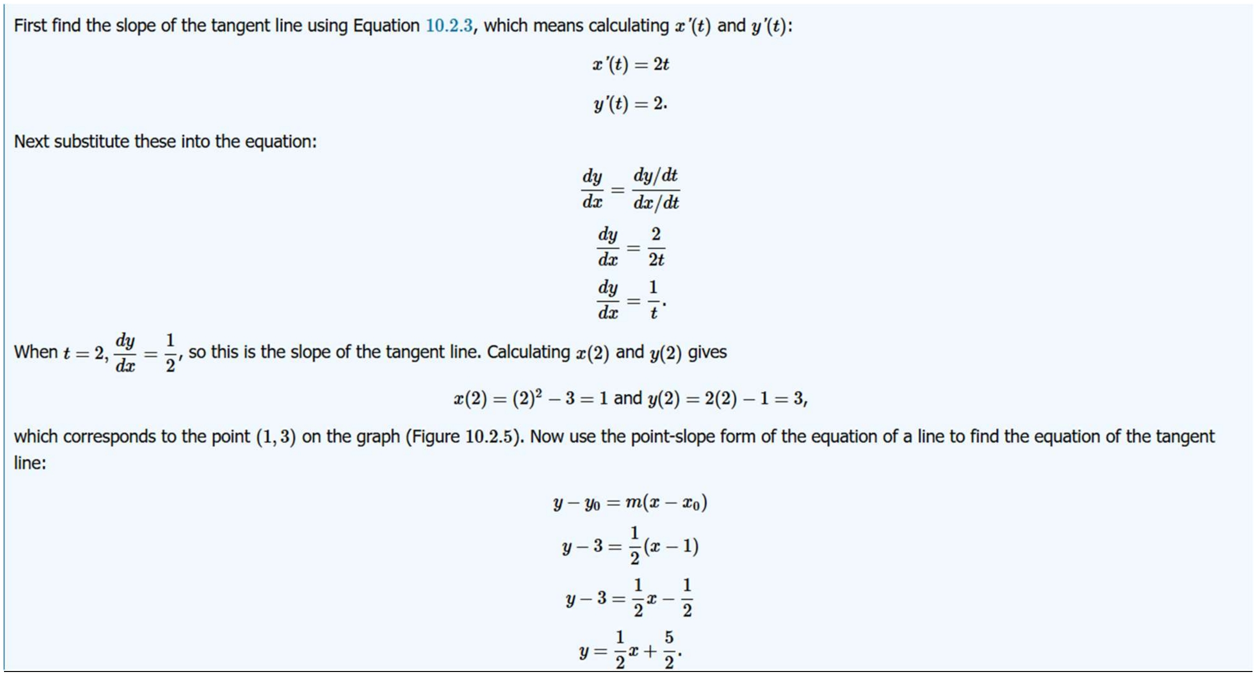
If a curve is not smooth at $t = t_0$, it means that $x'(t_0) = y'(t_0) = 0$ as defined. This, in turn, means that rate of change f x (and y) is 0; that is, at that instant, neither x nor y is changing. If the parametric equations describe the path of some bject, this means the object is at rest at t_0. An object at rest can make a "sharp" change in direction, whereas moving bjects tend to change direction in a "smooth" fashion.



Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4$$

when $t = 2$.



First find the slope of the tangent line using Equation 10.2.3, which means calculating $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$

$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

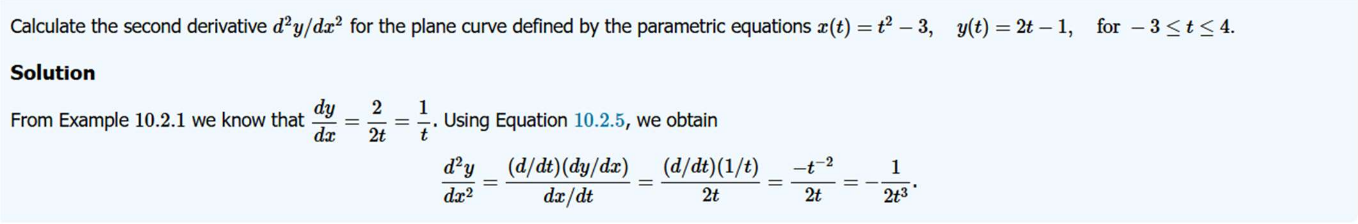
which corresponds to the point $(1, 3)$ on the graph (Figure 10.2.5). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$y - y_0 = m(x - x_0)$$

$$y - 3 = \frac{1}{2}(x - 1)$$

$$y - 3 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{5}{2}.$$



Calculate the second derivative d^2y/dx^2 for the plane curve defined by the parametric equations $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4.$

Solution

From Example 10.2.1 we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using Equation 10.2.5, we obtain

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

2. Consider the parametric curve $x = 4cos(3t)$, $y = 5sin(3t)$.

A. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{5 * 3cos(3t)}{-4 * 3sin(3t)} = \frac{-5}{4}cot(3t).$$

B. Find the equation of the tangent line to the curve at $t = \frac{\pi}{12}$.

$$m = \frac{-5}{4}cot(3\frac{\pi}{12}) = \frac{-5}{4}cot(\frac{\pi}{4}) = \frac{-5}{4}.$$

at $t = \frac{\pi}{12}$

$$x_0 = 4cos(\frac{3\pi}{12}) = 4cos(\frac{\pi}{4}) = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}$$

and

$$y_0 = 4sin(\frac{3\pi}{12}) = 4cos(\frac{\pi}{4}) = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}$$

So,

$$y - 2\sqrt{2} = \frac{-5}{4}(x - 2\sqrt{2})$$

$$\begin{aligned} y &= \frac{-5}{4}x + \frac{5\sqrt{2}}{2} + 2\sqrt{2} \\ &= \frac{-5}{4}x - \sqrt{2}(\frac{9}{2}) \end{aligned}$$

Find the arc length of the parametric curves below on the indicated intervals of the parameter.

A. $x = \cos(10t)$, $y = \sin(10t)$ for $0 \leq t \leq \pi$

$$L = \int_0^\pi \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = \int_0^\pi \sqrt{(-10sin(10t))^2 + (10cos(10t))^2}dt$$

$$= \int_0^\pi \sqrt{100(sin^2(10t) + cos^2(10t))}dt = 10\int_0^\pi dt = 10|_0^\pi = 10\pi$$

B. $x = 8t^{\frac{3}{2}}$, $y = 3 + (8-t)^{\frac{3}{2}}$ for $0 \leq t \leq 4$

$$L = \int_0^4 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = \int_0^4 \sqrt{(8\frac{3}{2}t^{\frac{1}{2}})^2 + (\frac{-3}{2}(8-t)^{\frac{1}{2}})^2}dt$$

$$= \int_0^4 \sqrt{144t + \frac{9}{4}(8-t)}dt = \int_0^4 \sqrt{\frac{567}{4}t + 18}dt$$

$$= \frac{4}{567}\frac{2}{3}(\frac{567}{4}t + 18)^{\frac{3}{2}}|_0^4 = 66.19$$

Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad 0 \le t \le 2\pi.$$

Solution

Using **Equation 7.3**, we have

$$\begin{aligned}
A &= \int_a^b y(t)x'(t)\,dt \\
&= \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt \\
&= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt \\
&= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right) dt \\
&= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos 2t}{2}\right) dt \\
&= \frac{3t}{2} - 2\sin t + \frac{\sin 2t}{4}\Big|_0^{2\pi} \\
&= 3\pi.
\end{aligned}$$

Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t) = r\cos t, \quad y(t) = r\sin t, \quad 0 \le t \le \pi.$$

This generates an upper semicircle of radius r centered at the origin as shown in the following graph.

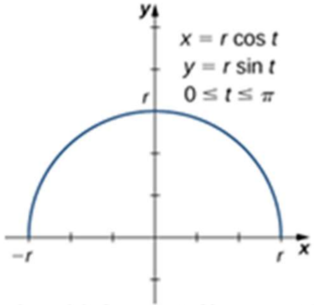


Figure 7.26 A semicircle generated by parametric equations.

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use **Equation 7.6**:

$$\begin{aligned}
S &= 2\pi\int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}dt \\
&= 2\pi\int_0^\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}dt \\
&= 2\pi\int_0^\pi r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt \\
&= 2\pi\int_0^\pi r\sin t\sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}dt \\
&= 2\pi\int_0^\pi r^2\sin t\,dt \\
&= 2\pi r^2(-\cos t|_0^\pi) \\
&= 2\pi r^2(-\cos\pi + \cos 0) \\
&= 4\pi r^2.
\end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.

E.5. The surface S is obtained by revolving the curve

$$C = \{(e^t - t, 4e^{t/2}) : t \in [0,1]\}$$

about the y-axis. Find the surface area of S.

We have

$$x(t) = e^t - t\,, \quad y(t) = 4e^{t/2}\,, \quad x'(t) = e^t - 1\,, \quad y'(t) = 2e^{t/2}\,.$$

By Formula 2 we have

$$\begin{aligned}
A =&\ 2\pi \int_0^1 (e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt \\
=&\ 2\pi \int_0^1 (e^t - t)\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\,dt \\
=&\ 2\pi \int_0^1 (e^t - t)\sqrt{e^{2t} + 1 + 2e^t}\,dt \\
=&\ 2\pi \int_0^1 (e^t - t)\sqrt{(e^t + 1)^2}\,dt = 2\pi \int_0^1 (e^t - t)(e^t + 1)\,dt \\
=&\ 2\pi \int_0^1 (e^{2t} - te^t + e^t - t)\,dt = 2\pi \left[\frac{e^{2t}}{2} - te^t + e^t + e^t - \frac{t^2}{2}\right]_0^1 \\
=&\ 2\pi \left[\frac{e^{2t}}{2} - te^t + 2e^t - \frac{t^2}{2}\right]_0^1 = 2\pi\left(\left(\frac{e^2}{2} - e + 2e - \frac{1}{2}\right) - \left(\frac{1}{2} + 2\right)\right) \\
=&\ \pi(e^2 + 2e - 6)\,,
\end{aligned}$$

Review for Sections 10.3, 10.4, 11.2, 11.3

10.3 and 10.4

4) $r = \cos(4\theta)$ is → even number

$2 \cdot 4 = 8$ petals

A) a line

[✓ a rose that has 8 petals]

C) a limacon (snails)

D) a rose that has 2 petals

E) a lemniscate (figure-8)

---

7) Convert the following equation from Cartesian form to Polar form
$y = x - 3$

$r\sin\theta = r\cos\theta - 3$

$\Rightarrow r(\cos\theta - \sin\theta) = 3$

$r = \dfrac{3}{\cos\theta - \sin\theta}$

A) $r = \dfrac{1}{\cos\theta - \sin\theta}$

B) $r = \dfrac{1}{3\sin\theta - \cos\theta}$

C) $r = \dfrac{1}{\sin\theta - 3\cos\theta}$

[✓ $r = \dfrac{3}{\cos\theta - \sin\theta}$]

E) $r = \dfrac{\cos\theta - \sin\theta}{2}$

8) Convert the point (7, - 7) from Cartesian to Polar coordinates. 8) ______

[✓ $\left(7\sqrt{2}, -\frac{\pi}{4}\right)$]

B) $\left(7, -\frac{\pi}{4}\right)$

C) $\left(7, \frac{\pi}{4}\right)$

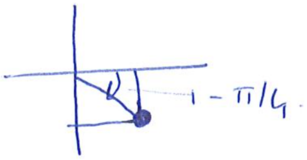
D) $\left(7\sqrt{2}, \frac{\pi}{4}\right)$

E) $\left(7\sqrt{2}, -\frac{7\pi}{4}\right)$

$r = \sqrt{49+49} = 7\sqrt{2}$

$\tan\theta = -1$

$\theta = -\pi/4$



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13) Convert the point $(-\sqrt{3}, -\sqrt{3})$ from Cartesian to Polar coordinates.

A) $\left(\sqrt{6}, \frac{\pi}{4}\right)$

B) $\left(3\sqrt{2}, \frac{\pi}{4}\right)$

[✓ $\left(\sqrt{6}, \frac{5\pi}{4}\right)$]

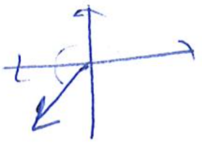
D) $\left(\sqrt{6}, -\frac{5\pi}{4}\right)$

E) $\left(\sqrt{3}, \frac{5\pi}{4}\right)$

$r = \sqrt{3+3} = 6$

$\tan\theta = 1$

$\theta = 5\pi/4.$



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14) Convert the following equation from polar to cartesian : 14) ______

r = 12 sinθ

A) $y^2 + (x - 6)^2 = 36$

[✓ $(y - 6)^2 + x^2 = 36$]

C) $y^2 + x^2 = 36$

D) $(y - 6)^2 + (x - 6)^2 = 36$

E) $y^2 + (x - 6)^2 = 6$

$\sqrt{x^2+y^2} = 12\frac{y}{r}$

$r = \sqrt{x2+y2}$

$\sqrt{x^2+y^2} = \frac{12y}{\sqrt{x^2+y^2}}$

$x^2+y^2 = 12y \Rightarrow x^2+y^2-12y = 0$

$x^2 + (y-6)^2 = 36.$

12) Bonus. Which one of the following is **false**:

A) $r = a \sin\theta$ is a circle

B) $r = a + \; b \cos\theta$ is a snail (Limacons)

C) $r = a + \; b \sin\theta$ is a snail (Limacons)

●) $r = 5\cos(5\theta)$ is a lemniscate

E) $r = a \cos\theta$ is a circle

(a) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \pi/3$.

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\,\sin\theta + (1 + \sin\theta)\cos\theta}{\cos\theta\,\cos\theta - (1 + \sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)(1 + 2\sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2\sin(\pi/3))}$$

$$= \frac{\frac{1}{2}\left(1 + \sqrt{3}\right)}{\left(1 + \sqrt{3}/2\right)\left(1 - \sqrt{3}\right)}$$

$$= \frac{1 + \sqrt{3}}{\left(2 + \sqrt{3}\right)\left(1 - \sqrt{3}\right)}$$

$$= \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1$$

Find the area enclosed by one loop of the four-leaved rose $r = \cos \underline{\underline{2}}\theta$. , $-\pi/4 < \theta < \pi/4$.

$$A_{loop} = \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2 \, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta \quad \underline{\underline{(7.2)}}$$

Side work

$$\cos^2 \underset{\overset{\Downarrow}{2\theta}}{\theta} = \frac{1}{2}(1+\cos 2\theta)$$

$$\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}\left(1+\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{\theta=-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4}+\frac{1}{4}\cdot 0\right)-\left(-\frac{\pi}{4}+\frac{1}{4}\cdot 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{4}+\left(+\frac{\pi}{4}\right)\right] = \frac{1}{4}\cdot\frac{2\pi}{4} = \frac{2}{16}\pi = \frac{\pi}{8}.$$

Set up the integral to

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

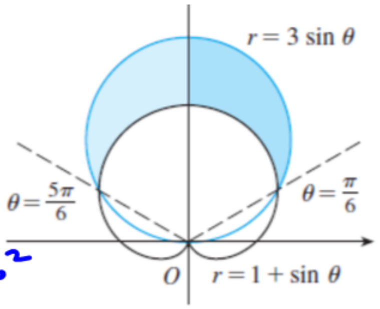


FIGURE 5

$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2\, d\theta$$

$(a+b)^2$

$a^2 + 2ab + b^2$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta \right] d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[8\sin^2\theta - 1 - 2\sin\theta \right] d\theta$$

Calculate

$$= (7.2) - \theta + 2\cos\theta \Big|_{\pi/6}^{5\pi/6}$$

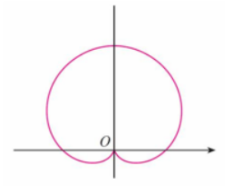
Set-up the integral to

Find the length of the cardioid $r = 1 + \sin\theta$.

$r' = \cos\theta$

Solution:

The cardioid is shown in Figure 8.



$r = 1 + \sin\theta$

Figure 8

$$L = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{\underset{=}{1} + 2\sin\theta + \underbrace{\sin^2\theta + \cos^2\theta}_{=1}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2(1+\sin\theta)}\, d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin\theta}\, d\theta$$

$\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\, d\theta$$

$$a^2 + b^2 + 2ab = (a+b)^2$$

Set-up

$$= \sqrt{2}\int_0^{2\pi} \cancel{\sqrt{\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^{\cancel{2}}}}\, d\theta = \boxed{\sqrt{2}\int_0^{2\pi} \left|\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right| d\theta}$$

11.2 and 11.3

Evaluate $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

geometric series

the first term = 1

common ratio = 1/2

(convergent)

$-1 < 1/2 < 1$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$$

$(r = 1/2)$

Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.

Geometric series

common ratio = $-1/3$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} = \frac{1}{1-(-\frac{1}{3})} = \frac{3}{4},$$

$-1 < -\frac{1}{3} < 1$, convergent

The first term = 1

$(r = -1/3)$

9) How many of the following p - series are **convergent** : 9) ______

$$\sum_{n=1}^{\infty} \frac{2}{n^5} \quad ✓$$

$$\sum_{n=1}^{\infty} 7n^{-3} = \frac{7}{n^3} \quad ✓$$

$$\sum_{n=1}^{\infty} 9n^{-1} = \frac{9}{n^1} \quad ✗$$

$$\sum_{n=1}^{\infty} \frac{-7}{n^4} \quad ✓$$

$$\sum_{n=1}^{\infty} \frac{12}{n^{-1}} \quad ✗$$

$$\sum_{n=1}^{\infty} 6n^5 = \frac{6}{n^{-5}} \quad ✗$$

$\sum \frac{1}{n^p}$ → $p > 1$, conv. ; → $p \leq 1$, divergent

A) 5 B) 1 C) 4 D) 2 [✓ E) 3]

---

21) $$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$$ is equal to

A) $\frac{19}{4}$

B) -1

C) $\frac{19}{3}$

D) 1

[✓ E) divergent]

$$\underbrace{5\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{1} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n}_{\text{geometric}}$$

$r = \frac{4}{3}$, divergent

EXAMPLE 11 Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$.

$$\underbrace{3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}} + \sum_{n=1}^{\infty}\frac{1}{2^n} \curvearrowright \text{geometric } r = 1/2$$

$$3\cdot 1 + \frac{1/2}{1-1/2} = 3+1 = 4$$

---

19) The sum of **the geometric series:** $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$ is

A) 3
B) 1
[✓ C) $\frac{1}{3}$]
D) $\frac{1}{2}$
E) divergent

$r = -1/2$, $-1 < r < 1$, convergent

$$\frac{\frac{1}{2}}{1-\left(-\frac{1}{2}\right)} = 1/3$$

---

20) Which one of the following is **true** for the following series

$$\sum_{n=1}^{\infty} 8\,\frac{n^3+1}{4n^3+12}$$

A) Converges to $\frac{1}{2}$
B) converges to $\frac{1}{4}$
C) converges to 2
D) converges to $\frac{1}{12}$
[✓ E) diverges]

Test for divergence

$\lim_{n\to\infty} a_n \neq 0$ or does not exist, $\sum a_n$ is divergent

$$\lim_{n\to\infty} 8\,\frac{n^3+1}{4n^3+12} = 8\lim_{n\to\infty}\frac{n^3+1}{4n^3+12}$$

$$= 8\cdot\frac{1}{4} = 2$$

Calculate the sum of the following series if it is convergent:

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots.$$

Hint:

$$\frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}.$$

1

$$S_n = \left(\frac{1}{1!} - \cancel{\frac{1}{2!}}\right) + \left(\cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}}\right) + + \left(\cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}}\right) + \cancel{\cdots} \cancel{+} \left(\cancel{\frac{1}{n!}} - \frac{1}{(n+1)!}\right)$$

$$S_n = 1 - \frac{1}{(n+1)!}$$

$$\lim_{n \to \infty} S_n = 1 - 0 = \boxed{1}$$

16) $\lim_{n \to \infty} 2\frac{(-1)^n}{2n+5}$ is equal to

A) 1
B) -1
C) 2
D) does not exist
[✓ E) 0]

if $\lim_{n \to \infty} |a_n| = 0 \Rightarrow \lim_{n \to \infty} a_n = 0.$

$$\lim_{n \to \infty} \left| \frac{\overbrace{2(-1)^n}^{-2,2}}{2n+5} \right| = 2 \lim_{n \to \infty} \underbrace{\frac{1}{2n+5}}_{0}$$

$$= 2 \cdot 0 = 0$$

37.41 Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

$f(x) = \frac{1}{x \ln x} \Rightarrow > 0$, cont, $f'(x) < 0$

(f is decreasing)

$\int_2^\infty \frac{1}{x \ln x} dx = \lim_{n \to \infty} \int_2^{n} \frac{du}{u}$

$u = \ln x$

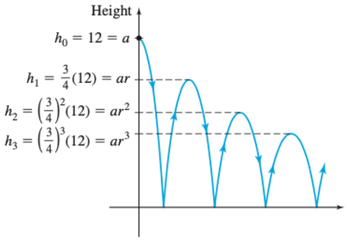
$du = \frac{1}{x} dx$

$= \lim_{n \to \infty} \ln |u| \Big|_2^{n} = \lim_{n \to \infty} \ln(\ln x) \Big|_2^{n}$

$= \lim_{n \to \infty} \ln(\ln n) - \ln(\ln 2) = \infty$

Since the improper integral is divergent, the series is divergent.

EXAMPLE 6 Using a Geometric Series with a Bouncing Ball



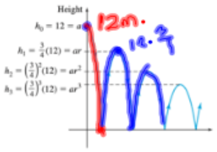
DF Figure 18

A ball is dropped from a height of 12 m. Each time it strikes the ground, it bounces back to a height three-fourths the distance from which it fell. Find the total distance traveled by the ball. See Figure 18.

Solution Let h_n denote the height of the ball on the nth bounce. Then

$h_0 = 12$

A ball is dropped from a height of 12 m. Each time it strikes the ground, it bounces back to a height three-fourths the distance from which it fell. Find the total distance traveled



DF Figure 18

$\text{Sum} = 12 + \overbrace{12 \cdot \frac{3}{4} + 12 \cdot \frac{3}{4}}^{2 \cdot 12 \cdot \frac{3}{4}}$

$\underbrace{12 \cdot \frac{3}{4} \cdot \frac{3}{4} + 12 \cdot \frac{3}{4} \cdot \frac{3}{4}}_{2 \cdot 12 \cdot \left(\frac{3}{4}\right)^2} + \cdots$

$= 12 + 2 \cdot 12 \left[\underbrace{\frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots}_{\text{Geom. Ser.}}\right]$

$= 12 + 24 \cdot \left[\frac{3/4}{1 - \frac{3}{4}}\right] \rightarrow \frac{3}{4} \cdot 4 = 3$

$= 12 + 24 \cdot 3 = 12 + 72 = 84$